BÉZOUT'S THEOREM AND AN IRRELEVANT POINT

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In the textbook I'm using for a first course in algebraic geometry, the proof of Bézout's theorem is awful. Looking around, I find an abundance of awful proofs. Here I give a proof due to Gurjar and Pathak that is not awful. I refer to their proof[1] for full details.

Theorem 1 (Bézout)**.** *Let k be an algebraically closed field. Let $f(X,Y,Z), g(X,Y,Z) \in k[X,Y,Z]$ be homogeneous polynomials of degrees m and n. Assume that the projective curves defined by f and g have no irreducible components in common. For each*

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[1]RV Gurjar and AK Pathak, *A short proof of Bézout's theorem in* $\mathbb{P}^2$. Communications in Algebra, 38:7, 2010, pp.2585–2587.

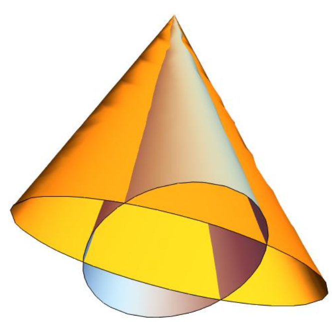

Figure 1. The four intersection points of two ellipses in the projective plane can be viewed as four lines that pass through the irrelevant point in three-dimensional affine space.

$p \in \mathbb{P}^2(k)$, let μ_p be the intersection multiplicity of $f = g = 0$ at p. Then

$$\sum_p \mu_p = mn.$$

That is, the number of points of intersection of the curves $f = 0$ and $g = 0$, counted with multiplicity, is mn.

In what follows, we will always count multiplicities, even if we do not constantly repeat the phrase *counted with multiplicity*. We will use the term multiset to refer to a set *with multiplicity*. The multiplicity is zero, except at points where the two curves intersect.

Recall that $\mathbb{P}^2$ is the set of lines in k^3. It is constructed by removing the origin 0, which is called the *irrelevant point*, and taking the quotient $\mathbb{P}^2 = (k^3 \setminus 0)/k^\times$. The proof will use the geometry of k^3, rather than $\mathbb{P}^2$, including the irrelevant point 0. In fact, the irrelevant point will be the key to the proof. The multiset of solutions of

$$f(X, Y, Z) = g(X, Y, Z) = 0, \tag{1}$$

can be viewed as a finite multiset of lines in k^3, with multiplicities μ_p. Thus, $\mu_p = 2$ indicates a doubled line associated with the point $p \in \mathbb{P}^2$, and $\mu_p = 3$ indicates a tripled line, etc.

We will need some basic facts about intersections of lines with lines in the plane. We say that two lines in a plane meet transversally if the two lines are distinct and are not parallel. In such a case, the lines meet at a single point with multiplicity one. Extending this to multiple lines, a multiset of m lines and a mutiset of n lines in a plane – such that each of the first group of lines meets each of the second group transversally – gives mn points of intersection, counted with multiplicity.

Similar results hold for intersections of lines with a plane in three-dimensional affine space. An intersection of a line with a plane is transversal if the line is not parallel to the plane and is not contained in the plane. In this case, the line and plane meet at a single point with multiplicity one. A multiset of ℓ lines – such that each line is transversal to the plane – meets the plane in ℓ points, counted with multiplicity.

Proof. Let L be the multiset of lines in k^3 defined by (1), and set $\ell := \sum_p \mu_p$. By construction, the number of lines in L counted with multiplicity is ℓ.

Using a linear transformation of k^3, we assume without generality that the monomial Z is not a factor of fg. We may also assume without loss of generality that no line of L lies in the plane $Z = 0$.

To prove Bézout's theorem, we count L a second way to show $\ell = mn$. Each line of L meets the plane $Z = 0$ transversally at the irrelevant point. Thus, the number

ℓ of lines is equal to the multiplicity of the intersection of L with the plane $Z = 0$. This intersection is given by the equations:

$$f(X, Y, Z) = g(X, Y, Z) = Z = 0,$$

or equivalently by the equations

$$f(X, Y, 0) = 0, \quad g(X, Y, 0) = 0, \tag{2}$$

in the plane $k^2 = \{(x, y, 0) \mid x, y \in k\}$.

We count solutions to (2). The polynomial $f(X, Y, 0)$ is homogeneous of degree m, by our assumption about the monomial Z not being a factor of f. By the fundamental theorem of algebra, the polynomial factors into a product of linear factors

$$f(X, Y, 0) = (a_1X + b_1Y)(a_2X + b_2Y) \cdots (a_mX + b_mY).$$

Hence $f(X, Y, 0) = 0$ defines a multiset L_1 of m lines in k^2 through the origin. By similar reasoning, $g(X, Y, 0) = 0$ defines a multiset L_2 of n lines in k^2 through the origin. The lines of L_1 meet the lines of L_2 transversally, by our assumptions about the plane $Z = 0$ not containing any line of L. Hence (2) gives the equations for the transversal intersection of m lines with n lines at the irrelevant point, for a total of mn intersections, counted with multiplicity. Thus, $\ell = mn$. □